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# Mappings commuting with the hat

by

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## 1. Introduction

In [1] the continuous map $f_2$ of the unit interval $[0,1]$ into itself, such that

$$f_2(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

called the hat, was studied. It was shown that $f_2$ commutes with all multihats $f_n$, where $f_n(x) = nx$ for $0 \leqslant x \leqslant \frac{1}{n}$, $f_n(x) = 2-nx$ if $\frac{1}{n} \leqslant x \leqslant \frac{2}{n}$, while $f_n$ is periodic with period $\frac{2}{n}$ (in particular, $f_0(x) = 0$ for all x). Furthermore it was proved that the semigroup of all $f_n$, $n = 0,1,2,\ldots$, is a maximal commutative semigroup: every continuous map of $[0,1]$ into itself commuting with all $f_n$ is one of the $f_n$.

Experiments lead us to the conviction that every continuous map commuting with $f_2$ is either a multihat $f_n$ or the map h such that h(x) equals the fixed point $\frac{2}{3}$ of $f_2$ (the function $f_0$ maps every x on the other fixed point of $f_2$, zero). We are as yet unable to stave this conviction with proof; but in this report we will show that in any case every open continuous map $g : [0,1] \to [0,1]$ commuting with $f_2$ is of necessity one of the multihats. In fact, an assumption much weaker than openness suffices: a continuous map g commuting with $f_2$ is a multihat as soon as $g(0) = 0$ while between two consecutive points where g equals zero there is located one and exactly one point where g takes the value 1.

As was pointed out to us by Z. Hedrlín, the results of [1] and of the present report give information about the Chebichev polynomials $T_n$, considered as transformations of the interval $[-1,1]$ into itself:

$$T_n(x) = \cos(n.\text{arc}\cos x).$$

For the map $\varphi(x) = \cos \pi x$ is a homeomorphism of $[0,1]$ onto $[-1,1]$, and $T_n = \varphi \circ f_n \circ \varphi^{-1}$, for $n = 0,1,2,\ldots$ . Hence it follows from the results of [1] that these polynomials constitute a maximal commutative semigroup of commutative mappings of $[-1,1]$ into itself, while the main result of the present report is equivalent to the following theorem.

Theorem. Every open continuous map $g : [-1,1] \to [-1,1]$ that commutes with the Chebychev polynomial $T_2$ is one of the Chebychev polynomials $T_n$.

## 2. Functions commuting with $f_2$

A map $g : [0,1] \to [0,1]$ is open if it maps open subsets of the topological space $[0,1]$ onto open subsets; i.e. if $g((a,b))$ is open for $0 \leq a < b \leq 1$, $g([0,a))$ is open for $0 < a \leq 1$, and $g((b,1])$ is open for $0 \leq b < 1$. From this readily follows:

Lemma 1. A continuous map $g : [0,1] \to [0,1]$ is open if and only if it is of the following type: there is a finite set of points $\{a_0, a_1, \ldots, a_n\}$, $0 = a_0 < a_1 < \ldots < a_n = 1$ such that for each $\nu$ the value $g(a_\nu)$ equals either 0 or 1, while $g \mid [a_\nu, a_{\nu+1}]$ is monotone, i.e. is a homeomorphism $[a_\nu, a_{\nu+1}] \to [0,1]$.

Remark. It follows that open maps coincide with full maps as defined by H. Cohen [2].

If we call a point $x$ with $g(x) = a$ an $a$-point of $g$, then it follows that 0-points and 1-points of an open continuous map $g$ alternate.

Any map $g$ commuting with $f_2$ maps the fixed point 0 of $f_2$ into another fixed point: $g(0) = 0$ or $g(\frac{2}{3})$. If moreover $g(0) = 0$, the only possibility is that $g(0) = 0$. Hence every open continuous map $g$ commuting with $f_2$ is of type I, where

Definition. A map $g : [0,1] \to [0,1]$ is of type I if there exists a finite set $\{a_1, a_2, a_3, \ldots, a_n, b_o, b_1, b_2, \ldots, b_m\}$ such that

(i) $0 = b_o < a_1 < b_1 < a_2 < b_2 < a_3 < \ldots$; either $a_n = 1$ or $b_m = 1$;

(ii) $g^{-1}(0) = \{b_o, b_1, \ldots, b_m\}$, $g^{-1}(1) = \{a_1, a_2, \ldots, a_n\}$.

In the remainder of this section g is assumed to be a continuous map of type I commuting with $f_2$. Simplifying the notation, we write f instead of $f_2$.

Lemma 2.
$$b_{2k} = 2b_k \quad , \quad \text{for} \quad 0 \leq 2k \leq m;$$
$$b_{2k-1} = 2a_k \quad , \quad \text{for} \quad 1 \leq 2k-1 \leq m.$$

Proof.

$0 = g(b_j) = gf(\frac{1}{2}b_j) = fg(\frac{1}{2}b_j) \Rightarrow g(\frac{1}{2}b_j) = 0$ or $1 \Rightarrow \frac{1}{2}b_j$ is either an $a_k \leq \frac{1}{2}$ or a $b_k \leq \frac{1}{2}$. Conversely, if x is either an $a_k \leq \frac{1}{2}$ or a $b_k \leq \frac{1}{2}$, then $g(2x) = gf(x) = fg(x) = 0$.

Lemma 3. If $g(1) = 0$, then $m = n$ and
$$a_{n-k} = 1 - a_{k+1} \quad \text{for } 0 \leq k \leq n-1;$$
$$b_{n-k} = 1 - b_k \quad \text{for } 0 \leq k \leq n.$$

Proof.

Clearly $m = n$. Let $x \leq \frac{1}{2}$. Then $g(x) \in \{0,1\} \Leftrightarrow g(2x) = fg(x) = 0 \Leftrightarrow g(1-x) \in \{0,1\}$.

Similarly:

Lemma 4. If $g(1) = 1$, then $m = n-1$ and
$$b_{n-k} = 1 - a_k, \quad \text{for } 1 \leq k \leq n.$$

Proposition 1. Let g be a continuous map of type I, commuting with f. Then

$$a_k = \frac{2k-1}{2n}, \quad b_k = \frac{2k}{2n} \quad (1 \leq k \leq n)$$

if $g(1) = 0$, and

$$a_k = \frac{2k-1}{2n-1}, \quad b_k = \frac{2k}{2n-1} \quad (1 \leq k \leq n)$$

if $g(1) = 1$.

Proof.

If $n = 1$, the assertions are trivial. Hence assume $n \geq 2$. We distinguish four cases, according to the parity of n and of $g(1)$.

Case 1: $g(1) = 0$, n even.

The inhomogeneous linear system of equations

$$(1) \qquad \begin{aligned} x_k + x_{n-k} &= 1 \qquad (1 \leq k \leq \tfrac{n}{2}), \\ 2x_k - x_{2k} &= 0 \qquad (1 \leq k \leq \tfrac{n}{2}), \end{aligned}$$

admits the solution $x_k = b_k$ $(1 \leq k \leq n)$, by lemma's 2,3. But obviously the following is also a solution: $x_k = \frac{k}{n}$ $(1 \leq k \leq \frac{n}{2})$. Hence it suffices to show that the system (1) has a non-zero determinant, for then we must have: $b_k = \frac{k}{n}$, while $a_k = \frac{1}{2}b_{2k-1} = \frac{2k-1}{2n}$, by lemma 3.

The form of the matrix M of (1) is indicated on page 5.

If one subtracts the (n-k)-th column from the k-th one, for $1 \leq k \leq \frac{n}{2} - 1$, the first $n/_2$ rows are all left with exactly one non-zero entry. Moreover, the last column has only one non-zero entry to begin with. Hence if we next strike out the first $\frac{n}{2}$ rows, the last row, and the last $\frac{n}{2} + 1$ columns, we are left with a matrix $M_1$ the determinant of which differs from det(M) only by a factor $\pm 1$.

$$M = \left(\begin{array}{cccccc|cccccc}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \iddots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\hline
2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 2 & 0 & 0 & \cdots & 0 & 0 & -1
\end{array}\right)
\begin{array}{l} \updownarrow \, n/2 \\ \\ \updownarrow \, n/2 \end{array}$$

$$\longleftarrow n/2 \longrightarrow \qquad \longleftarrow n/2 \longrightarrow$$

$$M_1 = \begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & 0 \\
0 & 2 & 0 & -1 & \cdots & 0 \\
0 & 0 & 2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \iddots & \vdots \\
0 & 0 & 0 & 1 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 2
\end{pmatrix} \updownarrow \, n/2-1$$

The determinant of $M_1$ is easily seen to be equal to $2^{\frac{n}{2}-1}$ (reduce $M_1$ to triangular form); hence $\det(M) \neq 0$.

<u>Case 2</u>: $g(1) = 0$, $n$ odd.

The $(n-1) \times (n-1)$ inhomogeneous linear system

$$
\begin{aligned}
x_k + x_{n-k} &= 1 \qquad (1 \leq k \leq \tfrac{n}{2}-1) \\
2x_k - x_{2k} &= 0 \qquad (1 \leq k \leq \tfrac{n-1}{2})
\end{aligned}
\tag{2}
$$

admits the solutions $x_k = b_k$ $(1 \leq k \leq \frac{n-1}{2})$ and $x_k = \frac{k}{n}$ $(1 \leq k \leq \frac{n-1}{2})$. Using a reduction as in case 1 we easily find that the determinant of (2) differs only by a factor $\pm 1$ from the determinant of a matrix

$$
\begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & 0 \\
0 & 2 & 0 & 1 & \cdots & 0 \\
0 & 0 & 2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \ddots & \vdots \\
0 & 0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 2
\end{pmatrix}.
$$

Evaluating this matrix modulo 2, we see that its determinant is an odd integer, hence differs from 0.

<u>Case 3</u>: $g(1) = 1$, $n$ even.

Use the square system

$$
\begin{aligned}
x_k + x_{2n-k-1} &= 1 \qquad (1 \leq k \leq n-1); \\
2x_k - x_{n+2k-2} &= 0 \qquad (1 \leq k \leq \tfrac{n}{2}); \\
2x_{n+k-1} - x_{n+2k-1} &= 0 \qquad (1 \leq k \leq \tfrac{n}{2}-1).
\end{aligned}
\tag{3}
$$

One solution is given by $x_k = a_k$, $x_{n+k-1} = b_k$ $(1 \leq k \leq n-1)$, another by $x_k = \frac{2k-1}{2n-1}$, $x_{n+k-1} = \frac{2k}{2n-1}$ $(1 \leq k \leq n-1)$. The determinant of (3) is easily seen to differ only by a factor $\pm 1$ from the determinant of a matrix of the form

$$\begin{pmatrix} 2 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 2 & \cdots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 2 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}.$$

If one considers the coefficients of this matrix modulo 2 one arrives at a matrix each row and column of which has only one non-zero entry, this entry being equal to $\pm 1$. The determinant of such a matrix equals $\pm 1$, hence the determinant of (3) is odd, thus certainly non-zero, proving that $a_k = \frac{2k-1}{2n-1}$, $b_k = \frac{2k}{2n-1}$, for $1 \leq k \leq n-1$.

Case 4: $g(1) = 1$, $n$ odd.

The proof is similar to the one in the cases 2 and 3, using the system

$$\begin{aligned} x_k + x_{2n-k-1} &= 1 && (1 \leq k \leq n-1); \\ 2x_k - x_{n+2k-2} &= 0 && (1 \leq k \leq \tfrac{n-1}{2}); \\ 2x_{n+k-1} - x_{n+2k-1} &= 0 && (1 \leq k \leq \tfrac{n-1}{2}), \end{aligned}$$

which is satisfied by $x_k = a_k$, $x_{n+k-1} = b_k$ $(1 \leq k \leq n-1)$.

Proposition 2. Let n be a natural number, and let g be a continuous map $[0,1]\to[0,1]$ commuting with $f_2$ such that

$$g^{-1}(0) = \left\{ \frac{2k}{n} : k \text{ integer and } 0 \leq k \leq \frac{n}{2} \right\};$$

$$g^{-1}(1) = \left\{ \frac{2k-1}{n} : k \text{ integer and } 0 \leq k \leq \frac{n+1}{2} \right\}.$$

Then $g = f_n$.

Proof.

Suppose $g(x) = f_n(x)$ for all $x \in [0,1]$ such that $2^m f_n(x)$ is an integer (by assumption this is true if $m = 0$). We will show that also $g(x) = f_n(x)$ if $2^{m+1} f_n(x)$ is integer.

Indeed, let $f_n(x) = \frac{k}{2^{m+1}}$, k integer. If $x \leq \frac{1}{2}$, then $f_n(2x) = f_n f_2(x) = f_2 f_n(x) = \frac{k'}{2^m}$, k' integer; hence $f_n(2x) = g(2x)$, which implies

$$f_2 f_n(x) = f_2 g(x). \tag{5}$$

Similarly we find that (5) holds for $x \geq \frac{1}{2}$. It then follows that either $f_n(x) = g(x)$ - in which case we are ready - or $f_n(x) = 1 - g(x)$. But this is impossible except if $f_n(x) = g(x) = 1$, as one verifies by considering the largest $x_1 < x$ and the least $x_2 > x$ such that $2^m f_n(x_i)$ is integer (taking into account the continuity of g).

Hence g and $f_n$ coincide on a dense subset of $[0,1]$. As both maps are continuous, it follows that $g = f_n$.

## 3. The main results

From propositions 1 and 2 we conclude at once:

Theorem 1. Every continuous map $g : [0,1]\to[0,1]$ of type I that commutes with $f_2$ is a multihat $f_n$.

As every open continuous map that commutes with $f_2$ is of type I, we have in particular:

Theorem 2. Every open continuous map $g : [0,1] \to [0,1]$ that commutes with $f_2$ is a multihat $f_n$.

By the observations made in the introduction, this theorem 2 is equivalent to

Theorem 3. Every open continuous map $g : [-1,1] \to [-1,1]$ that commutes with the Chebychev polynomial $T_2$ is itself a Chebychev polynomial $T_n$.

Remark. It seems that the methods of this report can be generalized to functions of type I that commute with $f_3$. However, we do not see at the moment how to arrive at a general description that fits all $f_n, n \geq 2$.

In any case it seems probable to us that the results of this note hold true more generally, i.e. that every continuous function of type I, in particular every open continuous map, that commutes with a multihat $f_n$, $n \geq 2$, is itself a multihat. This would mean that every open continuous map commuting with a Chebychev polynomial $T_n$, $n \geq 2$, is itself a Chebychev polynomial $T_n$.

## References

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2. H. COHEN, On fixed points of commuting functions. To appear in the Proc.Amer.Math. Society.